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SOME LOCALLY CONVEX SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS OVER A COMPLETELY REGULAR SPACE AND THEIR DUALS

BY

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ABSTRACT. The strict, superstrict and the β_F topologies are defined on a space A of continuous functions from a completely regular space into a Banach space E. Properties of these topologies are discussed and the corresponding dual spaces are identified with certain spaces of operator-valued measures. In case E is a Banach lattice, A becomes a lattice under the pointwise ordering and the strict and superstrict duals of A coincide with the spaces of all τ-additive and all σ-additive functionals on A respectively.

Introduction. The Riesz representation theorem says that any continuous linear functional F on the space of continuous real functions on a compact Hausdorff space X with the uniform topology must have the form $F(f) = \int_X f\, dm$ for some bounded regular Borel measure on X. This representation was extended later to other spaces, first in case X is locally compact and later by Aleksandrov [1] for continuous linear functionals on the space $C^b(X)$ of all bounded continuous real functions on a completely regular space. The representation was given by means of integrals with respect to members of the space $M(X)$ of all bounded, finitely-additive, regular with respect to zero sets, measures on the algebra generated by the zero sets. The σ-additive, τ-additive and tight linear functionals correspond to the σ-additive, τ-additive and tight members of $M(X)$ respectively (see Varadarajan [24]). Buck [4], for the locally compact case, and Sentilles [23], for the completely regular case, have defined the strict topologies on $C^b(X)$ which yield as dual spaces certain subspaces of $M(X)$. Several others like Hewitt [10], Bogdanowich [2], Wells [25], the author [12] and others have considered the problem of representation of linear functionals on spaces of continuous scalar-valued or vector-valued functions. In this paper we define certain locally convex topologies on spaces of continuous vector-valued functions on a completely regular space. We study some of the properties of these topologies and represent their duals with operator-valued measures on certain σ-algebras of subsets of X. The

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integration process employed is a generalization of the process of Aleksandrov to the vector case. It is one of the many integration processes defined by McShane [17].

1. **Definition and notation.** Throughout this paper X will denote a completely regular Hausdorff space and Y will be a Hausdorff compactification of X. We will denote by B the algebra of continuous real-valued functions f on X which have continuous extensions $\hat{f}$ to all of Y. Let E be a Banach space over the real field. We will denote by A the space of all continuous functions f from X into E which have continuous extensions $\hat{f}$ to all of Y. Let $C = \{\hat{f} : f \in A\}$. If $f \in A$ and $g \in B$, the function gf is defined on X by $(gf)(x) = g(x)f(x)$. For $s \in E$ we will denote also by s the element of A whose value at every x is equal to s.

We will consider on A various locally convex topologies.

(a) The uniform topology σ generated by the norm

$$f \longrightarrow \|f\| = \sup\{\|f(x)\| : x \in X\}.$$

(b) The topology k generated by the family seminorms p_K, K compact in X, where

$$p_K(f) = \|f\|_K = \sup\{\|f(x)\| : x \in K\}.$$

(c) The topology π generated by the family of seminorms P_x, $x \in X$, where

$$p_x(f) = \|f(x)\|.$$

Clearly all topologies π, k, σ are Hausdorff and $\pi \leqslant k \leqslant \sigma$. Finally, if τ is a linear topology on A, then $(A, \tau)'$ denotes the topological dual of (A, τ).

2. **The strict and superstrict topologies on A.** Buck [4] defined the strict topology on the space of bounded continuous functions on a locally compact Hausdorff space. This topology has been studied later by several other authors. Recently Sentilles [23] defined the strict and superstrict topologies on the family of all bounded, continuous, real-valued functions on a completely regular Hausdorff space. In this section we will define the strict and superstrict topologies on the space A defined in §1. Our approach will be analogous to that of Sentilles. Several of our theorems will be generalizations of his results.

A subset Z of Y is called a zero set if $Z = f^{-1}\{0\}$ for some continuous real function f on Y. We will denote by Ω (Ω_1) the class of all closed (zero) subsets of Y which are disjoint from X. For a Q in Ω, let $B_Q = \{f \in B : \hat{f} = 0 \text{ on } Q\}$. It is not hard to see that B_Q is a Banach algebra (under the uniform norm) with an approximate identy of norm $\leqslant 1$.

Let $Q \in \Omega$. We will denote by β_Q the locally convex topology on A generated by the family of seminorms $f \longrightarrow \|gf\|$, $g \in B_Q$. The space (A, σ) is a Banach space and a B_Q-module since $gf \in A$ for every $g \in B_Q$ and every $f \in A$. The topology β_Q is the strict topology on A as defined by Sentilles [20]. Hence

β_Q is the finest locally convex topology on A which agrees with β_Q on norm bounded subsets of A by Sentilles [21, Theorem 2.2]. A convex balanced absorbent set W in A is a β_Q-neighborhood of zero iff given $r > 0$ there exists a β_Q-neighborhood V of zero such that $U_r \cap V \subset W$, where $U_r = \{f \in A : \|f\| \leqslant r\}$.

The strict topology $\beta = \beta(A)$ on A is defined to be the inductive limit of the topologies β_Q, $Q \in \Omega$. The superstrict topology β_1 is the inductive limit of the topologies β_Z, $Z \in \Omega_1$. By definition of the inductive limit topology (see Schaefer [19, p. 57]), a convex balanced absorbent subset W of A is a β (β_1) neighborhood of zero iff W is a β_Q-neighborhood of zero for each $Q \in \Omega$ $(Q \in \Omega_1)$.

THEOREM 2.1. $k \leqslant \beta \leqslant \beta_1 \leqslant \sigma$.

PROOF. It is clear that $\beta \leqslant \beta_1 \leqslant \sigma$. To prove that $k \leqslant \beta$ consider an arbitrary compact set K in X. We want to show that the set $W = \{f \in A : \|f\|_K \leqslant 1\}$ is a β-neighborhood of zero. Since W is convex balanced and absorbent it suffices to show that W is a β_Q-neighborhood of zero for every Q in Ω. So, let $Q \in \Omega$. Since K is compact, there exists $g \in B$ such that $g = 1$ on K and $\hat{g} = 0$ on Q. Then $g \in B_Q$ and $V = \{f \in A : \|gf\| \leqslant 1\} \subset W$. Since V is a β_Q-neighborhood of zero the result follows.

Since, for each $Q \in \Omega$, β_Q is the finest locally convex topology on A which agrees with β_Q on norm bounded subsets of A, it follows that β (β_1) is the finest locally convex topology τ on A which agrees with β (β_1) on norm bounded sets.

LEMMA 2.2. *Let π be a locally convex topology on A such that $\pi \leqslant \tau \leqslant \sigma$ and such that $(A, \tau)' = H$ is a norm closed subspace of $A' = (A, \sigma)'$. Then τ and σ have the same bounded sets.*

PROOF. Every σ-bounded set is obviously τ-bounded. On the other hand, suppose that G is a τ-bounded subset of A. Then G is $\sigma(A, H)$ bounded. By our hypothesis H is a Banach space under the norm

$$\phi \longrightarrow \|\phi\| = \sup\{|\phi(f)| : f \in A,\ \|f\| \leqslant 1\}.$$

Each $f \in A$ defines a bounded linear functional T_f on H by $T_f(\phi) = \phi(f)$. Since G is $\sigma(A, H)$ bounded, we have $\sup\{|T_f(\phi)| : f \in G\} < \infty$ for each $\phi \in H$. By the principle of uniform boundedness there exists $K > 0$ such that $\sup\{\|T_f\| : f \in G\} \leqslant K$. Let now $f \in G$ and $x \in X$. By the Hahn-Banach theorem there is a T in E', $\|T\| \leqslant 1$, $T(f(x)) = \|f(x)\|$. Define $\pi_x : A \longrightarrow R$, $\pi_x(g) = T(g(x))$. Then π_x is in H since $\pi_x \in (A, \pi)' \subset H$. Moreover $\|\pi_x\| \leqslant 1$. Thus $\|f(x)\| = \pi_x(f) = T_f(\pi_x) \leqslant K$. It follows that $\sup\{\|f\| : f \in G\} \leqslant K$ which completes the proof.

THEOREM 2.3. *Let τ be as in Lemma 2.2. The following are equivalent:*

(1) $\tau = \sigma$.

(2) *τ is normable.*

(3) *τ is metrizable.*

(4) *τ is bornological.*

(5) *τ is barrelled.*

PROOF. It is clear that (1) implies (2), (2) implies (3), and (3) implies (4). To prove that (4) implies (5), we first observe that the set $U_1 = \{f \in A : \|f\| \leqslant 1\}$ is convex balanced and absorbs every norm (and hence every τ) bounded set. By (4) U_1 is a τ-neighborhood of zero. It follows that $\tau = \sigma$. Since (A, σ) is a Banach space, (5) follows. Finally to prove that (5) implies (1) we observe that the set U_1 is π-closed and hence τ-closed. Since U_1 is also convex balanced and absorbent it follows that U_1 is a τ-neighborhood of zero and hence $\tau = \sigma$.

THEOREM 2.4. *The duals of the spaces (A, β) and (A, β_1) are norm closed subspaces of the dual of (A, σ).*

PROOF. Let $\{T_n\}$ be a sequence in $(A, \beta)'$. $T \in A'$, such that $\|T_n - T\| \longrightarrow 0$. Let $W = \{f \in A : |T(f)| \leqslant 1\}$. We need to show that W is a β-neighborhood of zero. Since W is convex balanced and absorbent it suffices to show that given $r > 0$ there exists a β-neighborhood V of zero such that $V \cap \{f \in A : \|f\| \leqslant r\} \subset W$. So, let $r > 0$. Choose n so that $\|T_n - T\| < 1/(2r)$. Let $V = \{f \in A : |T_n(f)| < ½\}$. Then V is a β-neighborhood of zero and $V \cap \{f \in A : \|f\| \leqslant r\} \subset W$. This proves the result for β. The proof for β_1 is similar.

COROLLARY 2.5. (a) *Theorem 2.3 holds for $\tau = \beta$ or β_1.*

(b) *β, β_1 and σ have the same bounded sets.*

THEOREM 2.6. *The topologies β and σ coincide iff X is compact.*

PROOF. Clearly $\beta = \sigma$ when X is compact. On the other hand assume that X is not compact. Then $Y \neq X$. Let $x \in Y - X$, $Q = \{x\}$. Let $g \in B_Q$ and set $V = \{f \in A : \|gf\| \leqslant 1\}$. Choose $s \in E$, $\|s\| = 2$ and set $F = \{y \in Y : |\hat{g}(y)| \geqslant ½\}$. There exists $h \in B$, $0 \leqslant h \leqslant 1$, such that $\hat{h} = 0$ on F and $\hat{h}(x) = 1$. Then the function $f = hs$ is in V but not in $U_1 = \{f \in A : \|f\| \leqslant 1\}$. Hence U_1 does not contain V. It follows that U_1 is not a β_Q-neighborhood of zero and hence $\beta \neq \sigma$.

3. **The topology β_F.** Let F be a collection of compact subsets of X satisfying the following two conditions:

(1) $\bigcup F = X$.

(2) F is directed, i.e. given G_1, G_2 in F there exists G in F containing both G_1 and G_2. We denote by τ_F the locally convex topology on A generated by the family of seminorms $\{\|\cdot\|_G : G \in F\}$, where $\|f\|_G = \sup\{\|f(x)\| : x \in G\}$. The topology $\beta_F = \beta_F(A)$ is defined to be the mixed topology $\gamma[\sigma, \tau_F]$ as defined by Wiweger [26]. By Wiweger we have $\tau_F \leqslant \beta_F \leqslant \sigma$ and that β_F is the finest locally

convex topology on A which agrees with τ_F on each norm bounded subset of A (see Wiweger [26, 2.2.2.]).

LEMMA 3.1. *Let $G_1, \ldots, G_n$ be in F, $\epsilon > 0$ and f in A. Then there exist g, h in A such that $f = g + h$, $h = 0$ on each G_i, and* $\|g\| \leqslant \epsilon + \max\{\|f\|_{G_i}: 1 \leqslant i \leqslant n\}$.

PROOF. Let $G = \bigcup G_i$ and set $d = \epsilon + \|f\|_G$. Then $G \subset V = \{x \in Y: |\hat{f}(x)| < d\}$. Since G is compact and V open there exists $h_0 \in B$, $0 \leqslant h_0 \leqslant 1$, $\hat{h}_0 = 1$ on G, $\hat{h}_0 = 0$ on $Y - V$. Set $g = fh_0$ and $h = f(1 - h_0)$. Then g and h satisfy the requirements.

COROLLARY 3.2. *The sets of the form $\bigcap_{i=1}^{\infty}\{f \in A : \|f\|_{G_i} \leqslant a_i\}$, where $0 < a_i \longrightarrow \infty$ and $G_i \in F$, constitute a β_F-neighborhood base at zero.*

PROOF. It follows from the preceding lemma and from Wiweger's Theorem 3.1.1.

We will next give an alternative description of β_F. Denote by $B_0(F)$ the collection of all bounded real-valued functions f on X with the property that given $\epsilon > 0$ there exists G in F such that $\{x \in X: |f(x)| \geqslant \epsilon\} \subset G$. We define on A the locally convex topology $\tau(F)$ generated by the family of seminorms P_g, $g \in B_0(F)$, where

$$P_g(f) = \|gf\| = \sup\{\|g(x)f(x)\| : x \in X\}.$$

LEMMA 3.3. *The topologies $\tau(F)$ and τ_F coincide on each σ-bounded subset of the space A.*

PROOF. Let $r > 0$ and set $U_r = \{f \in A : \|f\| \leqslant r\}$. Suppose that V is a subset of U_r which is closed with respect to the relative topology of $\tau(F)$ on U_r. We will show that V is closed with respect to the τ_F relative topology on U_r. Indeed, let $f \in U_r$ be in the τ_F closure of V. Let $g \in B_0(F)$. Given $\epsilon > 0$ there exists G in F such that $|g(x)| < \epsilon/(2r)$ if x is not in G. Choose h in V with $\|f - h\|_G < \epsilon/\|g\|$. Then $\|g(f - h)\| \leqslant \epsilon$. This proves that f is in the $\tau(F)$ closure of V and hence in V. It follows that $\tau(F)|_{U_r} \leqslant \tau_F|_{U_r}$. On the other hand $\tau_F \leqslant \tau(F)$ because the characteristic function of any set in F belongs to $B_0(F)$. We conclude that $\tau(F)$ and τ_F agree on U_r. The result follows.

THEOREM 3.4. *The topologies $\tau(F)$ and β_F coincide.*

PROOF. Since β_F is the finest locally convex topology on A which agrees with τ_F on norm bounded subsets of A, we have $\tau(F) \leqslant \beta_F$ by 3.3. On the other hand, let $W = \bigcap_{i=1}^{\infty}\{f \in A : \|f\|_{G_i} \leqslant a_i\}$ where $0 < a_i \longrightarrow \infty$ and $G_i \in F$. Define $g = \sup_i a_i^{-1}\chi_{G_i}$ (χ_{G_i} is the characteristic function of G_i). Then $g \in B_0(F)$. Moreover, $\{h \in A : \|gh\| \leqslant 1\} \subset W$. Hence W is a $\tau(F)$-neighborhood of zero. Now Corollary 3.2 completes the proof.

THEOREM 3.5. *The topology β_F is weaker than β.*

PROOF. Let W be a convex balanced absorbent β_F-neighborhood of zero. Let $Q \in \Omega$ and $r > 0$. Since β_F coincides with τ_F on U_r there exist G in F and $\delta > 0$ such that $0 = \{f \in A : \|f\|_G \leqslant \delta\} \cap U_r \subset W$. Choose $g \in B_Q$, $0 \leqslant g \leqslant 1, g = 1$ on G. Then $V \cap U_r \subset W$, where $V = \{f \in A : \|gf\| \leqslant \delta\}$. This proves that W is a β_Q-neighborhood of zero for each Q in Ω. The theorem is proved.

We will next identify the space $(A, \beta_F)'$. Let $L_F(A)$ denote the collection of all linear functionals T in A' such that $T(f_\alpha) \longrightarrow 0$ whenever $\{f_\alpha\}$ is a net in U_1 that converges to zero uniformly on each $G \in F$. We omit the proof of the following easily established

LEMMA 3.6. *$L_F(A)$ is a norm closed subspace of A'.*

LEMMA 3.7. *Let $\phi \in (A, \sigma)'$. Then ϕ is in $L_F(A)$ iff for each $\epsilon > 0$ there exist G in F and $\delta > 0$ such that $|\phi(f)| \leqslant \epsilon$ for all $f \in A$ with $\|f\| \leqslant 1$ and $\|f\|_G \leqslant \delta$.*

PROOF. The necessity of this condition is clear. To prove the sufficiency assume, by way of contradiction, that there exists $\epsilon > 0$ such that for each $G \in F$ and each $\delta > 0$ there exists $f = f_{(G,\delta)}$ in A, with $\|f\| \leqslant 1$ and $\|f\|_G \leqslant \delta$, such that $|\phi(f)| > \epsilon$. For $\alpha_1 = (G_1, \delta_1), \alpha_2 = (G_2, \delta_2)$ we write $\alpha_1 \geqslant \alpha_2$ iff $G_1 \supset G_2$ and $\delta_1 \leqslant \delta_2$. In that way we get a net $\{f_\alpha\}$ in U_1. Moreover, $f_\alpha \longrightarrow 0$ uniformly on each $G \in F$. Since $|\phi(f_\alpha)| > \epsilon$ for all α we arrive at a contradiction.

THEOREM 3.8. *The topological dual of the space (A, β_F) is the space $L_F(A)$.*

PROOF. Let $\phi \in L_F(A)$. Set $W = \{f \in A : |\phi(f)| \leqslant 1\}$. If $r > 0$, there exist $G \in F$ and $\delta > 0$ such that $|\phi(f)| \leqslant 1/r$ whenever $\|f\| \leqslant 1$, $\|f\|_G \leqslant \delta$. Let $V = \{f \in A : \|f\|_G \leqslant \delta r\}$. Then V is a τ_F-neighborhood of zero and $V \cap U_r \subset W$. This shows that W is a β_F-neighborhood of zero in view of the fact that β_F is the finest locally convex topology on A which agrees with τ_F on norm bounded subsets of A. It follows that ϕ is β_F-continuous. Conversely, assume that ϕ is in $(A, \beta_F)'$ and let $\epsilon > 0$. There exist G in F and $\delta > 0$ such that

$$\{f \in A : \|f\|_G \leqslant \delta\} \cap U_{1/\epsilon} \subset \{f \in A : |\phi(f)| \leqslant 1\}.$$

Thus $|\phi(f)| \leqslant \epsilon$ whenever $f \in U_1$ and $\|f\|_G \leqslant \delta\epsilon$. By Lemma 3.7, $\phi \in L_F(A)$.

COROLLARY 3.9. (a) *β_F and σ have the same bounded sets.*

(b) *Theorem 2.3 holds if we replace τ with β_F.*

4. **The dual spaces of (A, β), (A, β_1), and (A, β_F).** In this section we will represent the dual spaces of (A, β), (A, β_1) and (A, β_F) by means of integrals with respect to operator-valued measures. We will denote by Bo(X) and Bo(Y)

the σ-algebras of Borel subsets of X and Y respectively.

The σ-algebra of Baire subsets of Y will be denoted by $\mathrm{Ba}(Y)$, while the σ-algebra, of subsets of X, generated by the B-zero sets will be denoted by $\mathrm{Ba}(Z_B)$ (a subset Z of X is called a B-zero set if $Z = f^{-1}\{0\}$ for some $f \in B$).

Let Σ be a σ-algebra of sets and let $\Sigma_1 \subset \Sigma$. A bounded, countably-additive, real-valued, measure m on Σ is called regular with respect to Σ_1 if for every $G \in \Sigma$ and every $\epsilon > 0$ there exists $G_1 \in \Sigma_1$, contained in G, such that $|m(H)| < \epsilon$ for every $H \in \Sigma$ which is contained in $G - G_1$.

We denote by $M_\sigma(B)$ the space of all bounded, real-valued, countably-additive, regular with respect to the family of all B-zero sets, measures on $\mathrm{Ba}(Z_B)$. The space of all bounded, real-valued countably-additive, regular with respect to the family of zero sets in Y, measures on $\mathrm{Ba}(Y)$ will be denoted by $M_\sigma(Y)$. A regular Borel measure on $\mathrm{Bo}(X)$ $(\mathrm{Bo}(Y))$ is a bounded, countably-additive, real-valued, measure on $\mathrm{Bo}(X)$ $(\mathrm{Bo}(Y))$ which is regular with respect to the closed sets in $X(Y)$. A regular Borel measure m on $\mathrm{Bo}(X)$ is called τ-additive if $|m|(F_\alpha) \longrightarrow 0$ for each net $\{F_\alpha\}$ of closed sets in X which decreases to the empty set. In the case of Y, every regular Borel measure is τ-additive (see [23]).

Note. A regular Borel measure m on $\mathrm{Bo}(X)$ is τ-additive iff $|m|(Z_\alpha) \longrightarrow 0$ for every net $\{Z_\alpha\}$ of B-zero sets which decreases to the empty set. Indeed, assume that the condition is satisfied and let $\{G_\alpha\}$ be a net of closed sets decreasing to the empty set. Since the zero sets in Y form a base for the closed sets, it follows that the family of B-zero sets forms a base for the closed sets in X. Thus each G_α is an intersection of B-zero sets. Let

$$D = \{Z \subset X : Z \text{ a } B\text{-zero set}, Z \supset G_\alpha \text{ for some } \alpha\}.$$

Then D is directed (by inclusion) downwards to the empty set. By hypothesis, given $\epsilon > 0$, there exists $Z \in D$ with $|m|(Z) < \epsilon$. Let α_0 be such that $G_{\alpha_0} \subset Z$. Now for each $\alpha \geqslant \alpha_0$ we have $|m|(G_\alpha) \leqslant |m|(Z) < \epsilon$ which proves that $|m|(G_\alpha) \longrightarrow 0$.

We will denote by $M_\tau(X)$ and $M_\tau(Y)$, respectively, the spaces of all τ-additive regular Borel measures on X and Y.

Let $M_\sigma(\mathrm{Ba}(Z_B), E')$ denote the set of all functions $m : \mathrm{Ba}(Z_B) \longrightarrow E'$ with the following two properties:

(1) For each $s \in E$, the function $ms : \mathrm{Ba}(Z_B) \longrightarrow R$, $(ms)(F) = m(F)s$, is in $M_\sigma(B)$.

(2) $|m|(X) < \infty$, where $|m|$ is defined on $\mathrm{Ba}(Z_B)$ by $|m|(G) = \sup|\Sigma m(F_i)s_i|$, the supremum being taken over all finite partitions $\{F_i\}$ of G into sets in $\mathrm{Ba}(Z_B)$ (we will refer to such a partition as a $\mathrm{Ba}(Z_B)$-partition) and all finite collections $\{s_i\} \subset E$ with $\|s_i\| \leqslant 1$. We define $M_\tau(\mathrm{Bo}(X), E')$ as we did $M_\sigma(\mathrm{Ba}(Z_B), E')$ by replacing $\mathrm{Ba}(Z_B)$ with $\mathrm{Bo}(X)$ and $M_\sigma(B)$ with $M_\tau(X)$. The

spaces $M_\sigma(\mathrm{Ba}(Y), E')$ and $M_\tau(\mathrm{Bo}(Y), E')$ are defined analogously. For m in any one of the above spaces, we define its norm $\|m\|$ by $\|m\| = |m|(X)$ or $|m|(Y)$ (depending on the σ-algebra on which m is defined).

THEOREM 4.1. *If* $m \in M_\sigma(\mathrm{Ba}(Z_B), E')$, *then* $|m| \in M_\sigma(B)$.

PROOF. It is easy to see that $|m|$ is a bounded, monotone, finitely-additive, set function on $\mathrm{Ba}(Z_B)$. To prove the regularity, consider a $G \in \mathrm{Ba}(Z_B)$ and let $\epsilon > 0$ be given. By the definition of $|m|(G)$, there exist a finite $\mathrm{Ba}(Z_B)$-partition $\{F_i\}$ of G and $s_i \in E$, with $\|s_i\| \leqslant 1$, such that $\Sigma m(F_i)s_i > |m|(G) - \epsilon$. By the regularity of each ms_i, there are B-zero sets $Z_i \subset F_i$ such that $\Sigma m(Z_i)s_i > |m|(G) - \epsilon$. The B-zero set $Z = \bigcup Z_i$ is contained in G. Moreover, $|m|(Z) \geqslant \Sigma m(Z_i)s_i > |m|(G) - \epsilon$ which proves the regularity of $|m|$.

To finish the proof it remains to show that $|m|$ is countably-additive. To this end, consider a sequence $\{F_i\}$ of disjoint members of $\mathrm{Ba}(Z_B)$ and let $G = \bigcup_i F_i$. Since $|m|$ is monotone and finitely-additive, we have

$$|m|(G) \geqslant |m|\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} |m|(F_i)$$

for all n and hence $|m|(G) \geqslant \Sigma_{i=1}^\infty |m|(F_i)$. On the other hand, let $\epsilon > 0$ be given. There exist a $\mathrm{Ba}(Z_B)$-partition $G_1, \ldots, G_N$ of G, and $s_i \in E$, with $\|s_i\| \leqslant 1$, such that $\Sigma_1^N m(G_i)s_i > |m|(G) - \epsilon$. For each i we have $m(G_i)s_i = \Sigma_{n=1}^\infty m(G_i \cap F_n)s_i$. Moreover,

$$\sum_{n=1}^{\infty} \sum_{i=1}^{N} |m(G_i \cap F_n)s_i| \leqslant \sum_{n=1}^{\infty} |m|(F_n) \leqslant |m|(G).$$

Hence

$$|m|(G) - \epsilon < \sum_{i=1}^{N} m(G_i)s_i = \sum_{i=1}^{N} \sum_{n=1}^{\infty} m(G_i \cap F_n)s_i$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{N} m(G_i \cap F_n)s_i < \sum_{n=1}^{\infty} |m|(F_n) \leqslant |m|(G).$$

Since $\epsilon > 0$ was arbitrary, we conclude that $|m|(G) = \Sigma_{n=1}^\infty |m|(F_n)$. This completes the proof.

THEOREM 4.2. *If* $m \in M_\tau(\mathrm{Bo}(X), E')$, *then* $|m| \in M_\tau(X)$.

PROOF. Using an argument similar to that of Theorem 4.1, we show that $|m|$ is a regular Borel measure on $\mathrm{Bo}(X)$. It remains to show that $|m|$ is τ-additive. By the note at the beginning of §4, it suffices to show that $|m|(Z_\alpha) \longrightarrow 0$ for each net $\{Z_\alpha\}$ of B-zero sets which decreases to the empty set. So, let $\{Z_\alpha\}$ be such a net. For each α there exists a zero set $\hat{Z}_\alpha$ in Y such that $Z_\alpha = \hat{Z}_\alpha \cap X$. Define $\bar{m} : \mathrm{Bo}(Y) \longrightarrow E'$ by $\bar{m}(F) = m(F \cap X)$. For each $s \in E$, the function $\bar{m}s : \mathrm{Bo}(Y) \longrightarrow R$, $(\bar{m}s)(F) = (ms)(F \cap X)$ is in $M_\tau(Y)$ since ms is τ-additive. It now follows easily that $\bar{m} \in M_\tau(\mathrm{Bo}(Y), E')$ and that $|\bar{m}|(F) = |m|(F \cap X)$ for each Borel set F in Y. Let D denote the collection of all subsets Z of Y which are intersections of a finite number of $\hat{Z}_\alpha$'s. Then D is directed downwards to $G = \bigcap \hat{Z}_\alpha$. Hence $|\bar{m}|(G) = \lim_{Z \in D} |\bar{m}|(Z)$. Since $G \cap X = \emptyset$ we have $|m|(G) = 0$. Therefore, given $\epsilon > 0$ there exists a $Z = \hat{Z}_{\alpha_1} \cap \cdots \cap \hat{Z}_{\alpha_n}$ in D such that $|\bar{m}|(Z) < \epsilon$. Now, if $\alpha \geqslant \alpha_1, \ldots, \alpha_n$, then

$$|m|(Z_\alpha) \leqslant |m|(Z_{\alpha_1} \cap \cdots \cap Z_{\alpha_n}) = |\bar{m}|(Z) < \epsilon.$$

This proves that $\lim |m|(Z_\alpha) = 0$ and the proof is complete.

We have analogous theorems for the elements in the spaces $M_\sigma(\mathrm{Ba}(Y), E')$ and $M_\tau(\mathrm{Bo}(Y), E')$.

Next we will define integrals with respect to measures belonging to one of the spaces defined above. The integration process which we will employ is a generalization, to the vector case, of the process of Aleksandrov. It is one of the many integration processes defined by McShane [17].

Let $m \in M_\sigma(\mathrm{Ba}(Z_B), E')$, $G \in \mathrm{Ba}(Z_B)$, and $f \in A$. Consider the collection D of all $\alpha = \{F_1, \ldots, F_n; x_1, \ldots, x_n\}$ where $\{F_i\}$ is a $\mathrm{Ba}(Z_B)$ partition of G and $x_i \in F_i$. For $\alpha_1, \alpha_2 \in D$ we write $\alpha_1 \geqslant \alpha_2$ iff the partition of G in α_1 is a refinement of the partition of G in α_2. Then $(D, \geqslant)$ is a directed set. For each $\alpha = \{F_1, \ldots, F_n; x_1, \ldots, x_n\}$ in D we define $w_\alpha = \Sigma\, m(F_i) f(x_i)$. We will show that $\{w_\alpha\}$ is a Cauchy net in R and hence convergent. Indeed, let $\epsilon > 0$ be given. We may assume, without loss of generality, that $\|m\| \leqslant 1$. For each $x \in X$, let $V_x = \{y \in X : \|f(x) - f(y)\| < \epsilon\}$. Then V_x is a B-cozero set and hence in $\mathrm{Ba}(Z_B)$: If $W = \{s \in E : \|s\| < \epsilon\}$, then $V_x = f^{-1}(f(x) + W)$. Since $f(X)$ is totally bounded, there are $x_1, \ldots, x_N$ in X such that $f(X) \subset \bigcup_{i=1}^N (f(x_i) + W)$. Thus $X = \bigcup_{i=1}^N V_{x_i}$. Let $G_i' = V_{x_i} \cap G$. Define $G_1 = G_1'$ and $G_{n+1} = G_{n+1}' - \bigcup_i^n G_i$, $n = 1, \ldots, N-1$. Keeping those G_i which are not empty we get a $\mathrm{Ba}(Z_B)$ partition $\{F_1, \ldots, F_n\}$ of G with the property that $\|f(x) - f(y)\| \leqslant 2\epsilon$ if x, y are in the same F_i. Pick $x_i \in F_i$ and let $\alpha_0 = \{F_1, \ldots, F_n; x_1, \ldots, x_n\}$. If α_1, α_2 are in D with $\alpha_1, \alpha_2 \geqslant \alpha_0$, then

$$|w_{\alpha_1} - w_{\alpha_2}| \leqslant |w_{\alpha_1} - w_{\alpha_0}| + |w_{\alpha_0} - w_{\alpha_2}| \leqslant 2\epsilon |m|(G) + 2\epsilon |m|(G) \leqslant 4\epsilon.$$

This proves that the net $\{w_\alpha\}$ is a Cauchy net and hence convergent. We define $\int_G f\, dm = \lim w_\alpha$. It can be shown easily that, for disjoint F_1 and F_2 in $\mathrm{Ba}(Z_B)$ and $G = F_1 \cup F_2$, we have $\int_G f\, dm = \int_{F_1} f\, dm + \int_{F_2} f\, dm$. Moreover we have the following easily established

LEMMA 4.3. (a) *The map* $f \longrightarrow \int_G f\, dm$ *is linear on* A.

(b) $$\left|\int_G f\, dm\right| \leqslant \int_G \|f(x)\|\, d|m|(x) \leqslant \|f\|\, |m|(G) \quad \textit{for all } f \in A.$$

We similarly define integrals of functions in A with respect to members of $M_\tau(\mathrm{Bo}(X), E')$.

LEMMA 4.4. *Let m be a bounded, real-valued, countably additive measure on* $\mathrm{Ba}(Z_B)$. *Then* $m \in M_\sigma(B)$.

PROOF. We first show that for any B-zero set Z and any $\epsilon > 0$ there exists a B-cozero set V, containing Z, such that $|m|(Z) > |m|(V) - \epsilon$. Indeed, if Z is a B-zero set, there exists $f \geqslant 0$ in B such that $Z = f^{-1}\{0\}$. For each positive integer n, we set $V_n = \{x : f(x) < 1/n\}$. Then V_n is a B-cozero set and the sequence $\{V_n\}$ decreases to Z. Since $|m|$ is countably additive, we have $|m|(Z) = \lim |m|(V_n)$ which implies our claim. Thus every B-zero set belongs to the family Σ of all subsets G of X with the following property: Given $\epsilon > 0$ there exist a B-zero set Z and a B-cozero set V, with $Z \subset G \subset V$, such that $|m|(V - Z) < \epsilon$. The family Σ is a σ-algebra which contains all B-zero sets and hence $\mathrm{Ba}(Z_B) \subset \Sigma$. This implies that m is regular with respect to the family of all B-zero sets. The lemma is proved.

LEMMA 4.5. *Let $m \in M_\tau(\mathrm{Bo}(X), E')$ and let μ denote the restriction of m to* $\mathrm{Ba}(Z_B)$. *Then $\mu \in M_\sigma(\mathrm{Ba}(Z_B), E')$ and $\int_G f\,dm = \int_G f\,d\mu$ for each $f \in A$.*

PROOF. In view of the preceding lemma, the restriction $ms|_{\mathrm{Ba}(Z_B)}$ belongs to $M_\sigma(B)$ for all $s \in E$. Now, it follows that $\mu \in M_\sigma(\mathrm{Ba}(Z_B), E')$. If we look at the proof of the existence of $\int_G f\,d\mu$ and $\int_G f\,dm$ we can see that $\int_G f\,dm$ and $\int_G f\,d\mu$ coincide.

Integrals of functions in C, with respect to members of $M_\sigma(\mathrm{Ba}(Y), E')$ and $M_\tau(\mathrm{Bo}(Y), E')$, are defined similarly.

LEMMA 4.6. *If $m_1, m_2 \in M_\tau(\mathrm{Bo}(Y), E')$ are such that $\int_Y \hat{f}\,dm_1 = \int_Y \hat{f}\,dm_2$ for all $f \in A$, then $m_1 = m_2$.*

PROOF. Let $s \in E$. For each $f \in B$, we have

$$\int_Y \hat{f}\,d(m_1 s) = \int_Y \hat{f}s\,dm_1 = \int_Y \hat{f}s\,dm_2 = \int_Y \hat{f}\,d(m_2 s).$$

By the uniqueness part of the Riesz representation theorem, we have $m_1 s = m_2 s$. This, being true for all $s \in E$, implies that $m_1 = m_2$.

For a proof of the following theorem see Wells [25].

THEOREM 4.7. *Let ϕ be a linear functional on C. Then ϕ is continuous with respect to the uniform norm topology iff there exists $m \in M_\tau(\mathrm{Bo}(Y), E')$ such that $\phi(\hat{f}) = \int_Y \hat{f}\,dm$ for all $\hat{f} \in C$. Moreover, $\|\phi\| = \|m\|$.*

If $m \in M_\tau(\mathrm{Bo}(Y), E')$ and $m_1 = m|_{\mathrm{Ba}(Y)}$ then $m_1 \in M_\sigma(\mathrm{Ba}(Y), E')$ and $\int_Y \hat{f}\,dm_1 = \int_Y \hat{f}\,dm$ for all $f \in A$. Furthermore, $\|m_1\| = \|m\|$. To prove the last equality, consider the linear map $\phi : C \longrightarrow R$, $\phi(\hat{f}) = \int_Y \hat{f}\,dm = \int_Y \hat{f}\,dm_1$. By 4.7

we have $\|\phi\| = \|m\|$. Also $\|\phi\| \leqslant \|m_1\|$ since $|\phi(\hat{f})| = |\int_Y \hat{f}\, dm_1| \leqslant \|\hat{f}\|\, \|m_1\|$. Since $\|m\| \geqslant \|m_1\|$, it follows that $\|\phi\| = \|m_1\| = \|m\|$. Moreover, the inequality $|m_1|(G) \leqslant |m|(G)$, together with $|m_1|(Y) = |m|(Y)$, implies that $|m_1| = |m|\,|_{\mathrm{Ba}(Y)}$.

Let now $\phi \in A'$. Define $\hat{\phi}: C \longrightarrow R$, $\hat{\phi}(\hat{f}) = \phi(f)$. Clearly $\hat{\phi} \in C'$. Let $m = \hat{m}_\phi$ be the element of $M_\tau(\mathrm{Bo}(Y), E')$ that corresponds to $\hat{\phi}$ by Theorem 4.7.

LEMMA 4.8. *For a $Q \in \Omega$, the following are equivalent*:

(1) $\phi \in (A, \beta_Q)'$.

(2) $|m|(Q) = 0$.

PROOF. (1) $\longrightarrow$ (2). By regularity it suffices to show that $m(G)s = 0$ for each closed set G in Y contained in Q and each $s \in E$, $\|s\| \leqslant 1$. So, let G be such a set and $s \in E$ with $\|s\| \leqslant 1$. There exists an open set O in Y containing G and such that $|ms|(O - G) < \epsilon$ ($\epsilon > 0$ arbitrary). Since ϕ is β_Q-continuous, there exist $g \in B_Q$ and $K > 0$ such that $|\phi(f)| \leqslant K$ for all $f \in A$ with $\|gf\| \leqslant 1$. Choose $n > 0$ so that $K/n < \epsilon$. Set

$$O_1 = \{x \in Y : |\hat{g}(x)| < 1/n\} \quad \text{and} \quad O_2 = O_1 \cap O.$$

Clearly $G \subset O_2$ and $|ms|(O_2 - G) < \epsilon$. Choose $h \in B$, $0 \leqslant h \leqslant 1$, $\hat{h} = 1$ on G and $\hat{h} = 0$ on the complement of O_2. Let $f = nhs$. Since $\|gf\| \leqslant 1$, we have $|\phi(hs)| \leqslant K/n < \epsilon$. But

$$|\phi(hs)| = \left| m(F)s + \int_{O_2 - G} \hat{h} s\, dm \right| \geqslant |m(F)s| - \epsilon.$$

Thus $|m(G)s| \leqslant 2\epsilon$ which proves that $m(G)s = 0$ and (2) follows.

(2) $\longrightarrow$ (1). Suppose that $|m|(Q) = 0$ and let $r > 0$. Choose an open set V in Y with $|m|(V) < 1/(2r), Q \subset V$. There exists $g \in B_Q$ such that $\hat{g} = 1$ on the complement of V. Set $W = \{f \in A : \|gf\| \leqslant 1/2\|m\|\}$. Then $W \cap U_r \subset H$ where $H = \{f \in A : |\phi(f)| \leqslant 1\}$ and $U_r = \{f \in A : \|f\| \leqslant r\}$. This shows that H is a β_Q-neighborhood of zero and hence ϕ is β_Q-continuous.

THEOREM 4.9. *Let $\phi \in A'$ and let $m \in M_\tau(\mathrm{Bo}(Y), E')$ be such that $\phi(f) = \int_Y \hat{f}\, dm$ for all $f \in A$. Then*:

(1) $\phi \in (A, \beta)'$ *iff* $|m|(Q) = 0$ *for all* $Q \in \Omega$.

(2) $\phi \in (A, \beta_1)'$ *iff* $|m|(Z) = 0$ *for all* $Z \in \Omega_1$.

PROOF. It follows from the preceding lemma and from the fact that ϕ is β-continuous iff ϕ is β_Q-continuous for all $Q \in \Omega$, and ϕ is β_1-continuous iff ϕ is β_Q-continuous for all $Q \in \Omega_1$.

Let now $\hat{m} \in M_\tau(\mathrm{Bo}(Y), E')$ be such that $|\hat{m}|(Q) = 0$ for all $Q \in \Omega$. By the regularity of $|\hat{m}|$, we have $|\hat{m}|(G) = 0$ for each Borel set G in Y disjoint from X. Define $m : \mathrm{Bo}(X) \longrightarrow E'$ by $m(G \cap X) = \hat{m}(G)$ for each G in $\mathrm{Bo}(Y)$. This gives us a well-defined function on Bo(X). The proof of the following is straightforward and we omit it.

LEMMA 4.10. (1) $m \in M_\tau(\mathrm{Bo}(X), E')$.
(2) $|m|(G \cap X) = |\hat{m}|(G)$ *for each* G *in* Bo(Y).
(3) $\int_X f\, dm = \int_Y \hat{f}\, d\hat{m}$ *for all* $f \in A$.

Similarly, if $\hat{m}_1 \in M_\sigma(\mathrm{Ba}(Y), E')$ is such that $|\hat{m}_1|(Z) = 0$ for each $Z \in \Omega_1$, then the function $m_1 : \mathrm{Ba}(Z_B) \longrightarrow E'$, $m_1(G \cap X) = \hat{m}_1(G)$ for all $G \in \mathrm{Ba}(Y)$, is well defined and the following is true.

LEMMA 4.11. (1) $m_1 \in M_\sigma(\mathrm{Ba}(Z_B), E')$.
(2) $|m_1|(G \cap X) = |\hat{m}_1|(G)$ *for each* G *in* Ba(Y).
(3) $\int_X f\, dm_1 = \int_Y \hat{f}\, d\hat{m}_1$ *for each* $f \in A$.

An element ϕ of the uniform dual B' of B is called τ-additive iff $\phi(f_\alpha) \longrightarrow 0$ for each net $\{f_\alpha\}$ in B which decreases pointwise to zero. Let $L_\tau(B)$ denote the collection of all τ-additive members of B.

LEMMA 4.12. *The map* $m \longrightarrow \phi$ *defined by the formula* $\phi(f) = \int f\, dm$ *for all* $f \in B$ *establishes an isomorphism between the spaces* $M_\tau(X)$ *and* $L_\tau(B)$.

PROOF. By LeCam [16, p. 214], every τ-additive member of B' has a unique extension to a τ-additive functional on the space $C^b(X)$ of all bounded continuous real-valued functions on X. By Varandarajan [24] and by Kirk [13, Theorem 1.12], the space of τ-additive functionals on $C^b(X)$ is isomorphic to the space $M_\tau(X)$ via the isomorphism $m \longrightarrow \phi$, $\phi(f) = \int f\, dm$ for all $f \in C^b(X)$. Hence the result follows.

LEMMA 4.13. *Let* $m \in M_\tau(X)$. *Define* $\bar{m}$ *on* Bo(Y) *by* $\bar{m}(G) = m(G \cap X)$. *Then* $\bar{m} \in \mathrm{Bo}(Y)$.

PROOF. By (4.12), the linear functional ϕ, defined on B by $\phi(f) = \int f\, dm$, is τ-additive. Define $\hat{\phi}$ on the space $C(Y) = \{\hat{f} : f \in B\}$ by $\hat{\phi}(\hat{f}) = \phi(f)$. Then $\hat{\phi}$ is in the uniform dual of $C(Y)$. By the Riesz representation theorem there exists $\mu \in M_\tau(Y)$ such that $\hat{\phi}(\hat{f}) = \int \hat{f}\, d\mu$ for all $f \in B$. By an argument similar to that employed by Knowles [14, Theorem 2.4], we show that $|\mu|(G) = 0$ for each Borel subset G of Y which is disjoint from X. Define m_1 on Bo(X) by $m_1(G \cap X) = \mu(G)$ for all $G \in \mathrm{Bo}(Y)$. It is easy to see that m_1 is a well-defined element of $M_\tau(X)$ and $\int_X f\, dm_1 = \int_Y \hat{f}\, d\mu = \phi(f) = \int_X f\, dm$ for all $f \in B$. By 4.12 we have $m = m_1$. Since for $G \in \mathrm{Bo}(Y)$, $\mu(G) = m_1(G \cap X) = m(G \cap X)$, it follows that $\hat{m} = \mu \in M_\tau(Y)$. This completes the proof.

LEMMA 4.14. *If* m *and* m_1 *are both in* $M_\sigma(B)$ *and if* $\int f\, dm = \int f\, dm_1$ *for all* $f \in B$, *then* $m = m_1$.

PROOF. Let Z be a B-zero set. There exists a sequence $\{f_n\}$ in B which decreases pointwise to the characteristic function $\chi_Z = g$ of Z. Thus

$$m(Z) = \int g\,dm = \lim\int f_n\,dm = \lim\int f_n\,dm_1 = m_1(Z).$$

The result now follows from the regularity of m and m_1.

LEMMA 4.15. *Let $m \in M_\sigma(B)$. Define $\bar{m}$ on $\mathrm{Ba}(Y)$ by $\bar{m}(G) = m(G \cap X)$ for all $G \in \mathrm{Ba}(Y)$. Then $\bar{m} \in M_\sigma(Y)$.*

PROOF. Let $\mu \in M_\tau(Y)$ be such that $\int_Y \hat{f}\,d\mu = \int_X f\,dm$ for all $f \in B$. Let $\mu_1 = \mu|_{\mathrm{Ba}(Y)}$. Then $\mu_1 \in M_\sigma(Y)$ and $\int_X f\,dm = \int_Y \hat{f}\,d\mu_1$ for all $f \in B$. Since the functional $f \longrightarrow \int f\,dm$ is σ-additive on B (i.e. $\int f_n\,dm \longrightarrow 0$ for each sequence $\{f_n\}$ in B which decreases pointwise to zero) it follows, as in the proof of Theorem 2.1 of Knowles [14], that $|\mu_1|(G) = 0$ for each Baire set G in Y which is disjoint from X. Define $m_1 : \mathrm{Ba}(Z_B) \longrightarrow R$, $m_1(G \cap X) = \mu_1(F)$ for each Baire set G in Y. Then m_1 is a well-defined member of $M_\sigma(B)$ and $\int f\,dm_1 = \int \hat{f}\,d\mu_1 = \int f\,dm$ for all $f \in B$. By Lemma 4.14 we have $m_1 = \mu_1$. Thus $m = \mu_1 \in M_\sigma(Y)$.

Now using 4.13 and 4.15 we easily get the following result.

LEMMA 4.16. *Let $m \in M_\tau(\mathrm{Bo}(X), E')$ and $m_1 \in M_\sigma(\mathrm{Ba}(Z_B), E')$. Define $\hat{m}$ and $\hat{m}_1$ on $\mathrm{Bo}(Y)$, respectively, by $\hat{m}(Q) = m(Q \cap X)$, $\hat{m}_1(G) = m_1(G \cap X)$. Then*:

(1) *$\hat{m} \in M_\tau(\mathrm{Bo}(Y), E')$ and $\hat{m}_1 \in M_\sigma(\mathrm{Ba}(Y), E')$.*

(2) *$|\hat{m}|(Q) = |m|(Q \cap X)$ for all $Q \in \mathrm{Bo}(Y)$, and $|\hat{m}_1|(Q) = |m_1|(G \cap X)$ for all $G \in \mathrm{Ba}(Y)$.*

(3) *$\int_X f\,dm = \int_Y \hat{f}\,d\hat{m}$ and $\int_X f\,dm_1 = \int_Y \hat{f}\,d\hat{m}_1$ for all $f \in A$.*

LEMMA 4.17. *Every $m \in M_\sigma(\mathrm{Ba}(Y), E')$ has a unique extension to a μ in $M_\tau(\mathrm{Ba}(Y), E')$.*

PROOF. Define ϕ on C by $\phi(\hat{f}) = \int_Y \hat{f}\,dm$. Then $\phi \in C'$. By 4.7 there exists a unique μ in $M_\tau(\mathrm{Bo}(Y), E')$ such that $\phi(\hat{f}) = \int \hat{f}\,d\mu$ for all $f \in A$. Let $\mu_1 = \mu|_{\mathrm{Ba}(Y)}$. We will show that $\mu_1 = m$. Indeed, let $s \in E$. Then $\mu_1 s$ and ms are both in $M_\sigma(Y)$. Moreover $\int \hat{f}\,d(ms) = \int \hat{f}s\,dm = \int \hat{f}s\,d\mu_1 = \int \hat{f}\,d(\mu_1 s)$ for all $f \in B$. It follows that $ms = \mu_1 s$ for all $s \in E$ and hence $m = \mu_1$. Since $\mu \in M_\sigma(\mathrm{Bo}(Y), E')$ the result follows.

Combining Lemmas 4.6, 4.17 and 4.16 we get

LEMMA 4.18. *If $m_1, m_2 \in M_\tau(\mathrm{Bo}(X), E')$ $[m_1, m_2 \in M_\sigma(\mathrm{Ba}(Z_B), E')]$ are such that $\int_X f\,dm_1 = \int_X f\,dm_2$ for all $f \in A$, then $m_1 = m_2$.*

We are now in a position to identify the dual spaces of (A, β), (A, β_1) and (A, β_F).

THEOREM 4.19. *Let* $\phi \in A'$. *Then*:

(1) ϕ *is* β*-continuous iff there exists* $m \in M_\tau(\mathrm{Bo}(X), E')$ *such that* $\int f\,dm = \phi(f)$ *for all* $f \in A$.

(2) ϕ *is* β_1*-continuous iff there exists* $m \in M_\sigma(\mathrm{Ba}(Z_B), E')$ *such that* $\phi(f) = \int f\,dm$ *for all* $f \in A$.

Furthermore, the m that corresponds to a β*-continuous* (β_1*-continuous*) *member* ϕ *of* A' *is unique and* $\|\phi\| = \|m\|$.

PROOF. (1) Suppose that ϕ is β-continuous. Let $\hat{m}$ be the element of $M_\tau(\mathrm{Bo}(Y), E')$ with the property that $\phi(f) = \int \hat{f}\,d\hat{m}$ for all $f \in A$. Define m on $\mathrm{Bo}(X)$ by $m(G \cap X) = \hat{m}(G)$ for all $G \in \mathrm{Bo}(Y)$. This gives us an element m of $M_\tau(\mathrm{Bo}(X), E')$ by 4.9 and 4.10. Moreover, by 4.10, $\int_X f\,dm = \int_Y \hat{f}\,d\hat{m} = \phi(f)$ for all $f \in A$. Also $\|\phi\| = \|\hat{m}\| = \|m\|$. Conversely, let $m \in M_\tau(\mathrm{Bo}(X), E')$ be such that $\phi(f) = \int f\,dm$ for all $f \in A$. Define $\hat{m}$ on $\mathrm{Bo}(Y)$ by $\hat{m}(G) = m(G \cap X)$. By 4.16, we have $m \in M_\tau(\mathrm{Bo}(Y), E')$ and $\int_X f\,dm = \int_Y \hat{f}\,d\hat{m}$ for all $f \in A$. Since $|\hat{m}|(Q) = |m|(Q \cap X) = 0$ for all $Q \in \Omega$, we have $\phi \in (A, \beta)'$ by 4.9. Finally the uniqueness of m follows from 4.18.

(2) The proof is similar to that of (1).

THEOREM 4.20. *For a* $\phi \in A'$ *the following are equivalent*:

(1) $\phi \in (A, \beta_F)'$.

(2) *There exists* $m \in M_\tau(\mathrm{Bo}(X), E')$ *such that*

(a) $\phi(f) = \int f\,dm$ *for all* $f \in A$,

(b) *given* $\epsilon > 0$ *there exists* $G \in F$ *with* $|m|(X - G) < \epsilon$.

PROOF. (2) $\longrightarrow$ (1). Let $\epsilon > 0$ be given. Choose G in F with $|m|(X - G) < \epsilon/2$. Let $\delta > 0$ be such that $2\delta\|m\| < \epsilon$. If $f \in A$, $\|f\| \leqslant 1$, $\|f\|_G \leqslant \delta$, then

$$|\phi(f)| \leqslant \left|\int_G f\,dm\right| + \left|\int_{X-G} f\,dm\right| \leqslant \delta\,|m|(G) + |m|(X - G) \leqslant \epsilon.$$

Hence $\phi \in (A, \beta_F)'$ by 3.7.

(1) $\longrightarrow$ (2). Since $\beta_F \leqslant \beta$, we have $\phi \in (A, \beta)'$. Hence there exists $m \in M_\tau(\mathrm{Bo}(X), E')$ such that $\phi(f) = \int f\,dm$ for all f in A. Define $\hat{m}$ on $\mathrm{Bo}(Y)$ by $\hat{m}(G) = m(G \cap X)$. By 4.16, $\hat{m} \in M_\tau(\mathrm{Bo}(Y), E')$. Let $\epsilon > 0$ be given. By 3.7 there exist G in F and $\delta > 0$ such that $|\phi(f)| \leqslant \epsilon_1 = \epsilon/3$ for all f in $W = \{h \in A : \|h\| \leqslant 1, \|h\|_G \leqslant \delta\}$. By the definition of $|\hat{m}|$ there exist a partition $F_1, \ldots, F_n$ of $Y - G$, into Borel sets, and $s_i \in E$, with $\|s_i\| \leqslant 1$, such that $\Sigma\hat{m}(F_i)s_i > |\hat{m}|(Y - G) - \epsilon_1 = |m|(X - G) - \epsilon_1$. There are closed sets G_i in Y, $G_i \subset F_i$, such that $\Sigma\hat{m}(G_i)s_i > |m|(X - G) - \epsilon_1$. Choose pairwise disjoint open sets V_i, $1 \leqslant i \leqslant n$, $G_i \subset V_i \subset Y - G$, such that $\Sigma\,|\hat{m}|(V_i - G_i) < \epsilon_1$. For each i, $1 \leqslant i \leqslant n$, choose h_i in B, $0 \leqslant h_i \leqslant 1$, $\hat{h}_i = 1$ on G_i and $\hat{h}_i = 0$ on $Y - V_i$. Let $f = \Sigma h_i s_i$. Then $f \in W$ and hence $|\phi(f)| \leqslant \epsilon_1$. Since

$$\phi(f) \leqslant \sum \int_{G_i} s_i \, d\hat{m} + \sum \int_{V_i - G_i} \hat{f} \, d\hat{m} \geqslant \sum \hat{m}(G_i) s_i - \epsilon_1 \geqslant |m|(X-G) - 2\epsilon_1,$$

it follows that $|m|(X - G) \leqslant 3\epsilon_1 = \epsilon$. The theorem is proved.

DEFINITION. A subset M_0 of $M_\tau(\mathrm{Bo}(X), E')$ is called F-tight if M_0 is norm bounded and given $\epsilon > 0$ there exists G in F with $|m|(X - G) \leqslant \epsilon$ for all m in M_0.

LEMMA 4.21. *Let $\phi \in (A, \beta)'$ and let m be the corresponding element of $M_\tau(\mathrm{Bo}(X), E')$. Let $G \in F$ and $\epsilon > 0$. The following are equivalent*:

(1) $|m|(X - G) \leqslant \epsilon$.

(2) *For all $f \in A$ with $\|f\| \leqslant 1$ and $\|f\|_G = 0$ we have $|\phi(f)| \leqslant \epsilon$.*

We omit the proof of this lemma since we can use an argument similar to that used in the implication (1) $\longrightarrow$ (2) of Theorem 4.20.

For $H \subset L_F(A)$, let $M_H = \{m_\phi : \phi \in H\} \subset M_\tau(\mathrm{Bo}(X), E')$ where m_ϕ is the measure that corresponds to ϕ.

THEOREM 4.22. *For $H \subset L_F(A)$ the following are equivalent*:

(1) *H is β_F-equicontinuous.*

(2) (a) *H is norm bounded.*

(b) *Given $\epsilon > 0$ there exists $G \in F$ such that $|\phi(f)| \leqslant \epsilon$ for all $\phi \in H$ and all $f \in A$ with $\|f\| \leqslant 1$ and $f = 0$ on G.*

(3) *M_H is F-tight.*

PROOF. By 4.21, (2) and (3) are equivalent.

(1) $\longrightarrow$ (2). (a) The set $U_1 = \{f \in A : \|f\| \leqslant 1\}$ is norm bounded and hence β-bounded. Since H^0 (= polar of H with respect to the pair $\langle L_F(A), A\rangle$) is a β_F-neighborhood of zero there exists $K > 0$ such that $U_1 \subset KH^0$. It follows that $\|\phi\| \leqslant K$ for all ϕ in H.

(b) Let $\epsilon > 0$ be given. Since ϵH^0 is a β_F-neighborhood of zero there exist G in F and $\delta > 0$ such that $W = \{f \in A : \|f\| \leqslant 1, \|f\|_G \leqslant \delta\} \subset \epsilon H^0$. Thus (b) follows.

(3) $\longrightarrow$ (1). Let $d = \sup\{\|m_\phi\| : \phi \in H\} = \sup\{\|m_\phi\| : \phi \in H\}$. Given $r > 0$ there exists $G \in F$ such that $|m_\phi|(X - F) \leqslant 1/(2r)$ for all $\phi \in H$. If $V = \{f \in A : \|f\|_G \leqslant 1/(2d)\}$, then $V \cap U_r \subset H^0$, where $U_r = \{f \in A : \|f\| \leqslant r\}$. This shows that H^0 is a β-neighborhood of zero and this completes the proof.

5. In this section we will assume that E is a Banach lattice. We write $f \geqslant g$ iff $f(x) \geqslant g(x)$ for all $x \in X$. Since the lattice operations are continuous, it is easy to verify that A, under the relation $\geqslant$, is a Banach lattice where for f, g in A we have

$$(f \wedge g)(x) = f(x) \wedge g(x),$$

$$(f \vee g)(x) = f(x) \vee g(x), \quad \text{and} \quad |f|(x) = |f(x)|$$

for all $x \in X$. For a $\phi \in A'$ the ϕ^+, ϕ^-, $|\phi|$ are the elements of A' which are defined on positive $f \in A$ by

$$\phi^+(f) = \sup\{\phi(g) : 0 \leqslant g \leqslant f\},$$
$$\phi^-(f) = -\inf\{\phi(g) : 0 \leqslant g \leqslant f\},$$
$$|\phi|(f) = \sup\{|\phi(g)| : |g| \leqslant f\}.$$

THEOREM 5.1. *Each of the spaces (A, β), (A, β_1) and (A, β_F) is locally solid.*

PROOF. Let W be a convex balanced β-neighborhood of zero. For each $Q \in \Omega$ there exists $g_Q \in B_Q$ such that $V_Q = \{f \in A : \|g_Q f\| \leqslant 1\} \subset W$. Each V_Q is clearly solid. Hence the set $V = \bigcup\{V_Q : Q \in \Omega\}$ is solid. By Peressini [18, p. 161], the convex balanced hull V_0 of V is solid. Since $V_0 \subset W$, the result follows for (A, β). The proof for (A, β_1) is similar. For the (A, β_F) we observe that the class of sets of the form $\bigcap_{i=1}^{\infty} \{f \in A : \|f\|_{G_i} \leqslant a_i\}$, where $0 < a_i \longrightarrow \infty$ and $G_i \in F$, consists of solid sets and is a β_F-base at zero.

DEFINITIONS. For a net $\{f_\alpha\}$ in A, we say that it decreases to zero, and write $f_\alpha \downarrow 0$, if for each $x \in X$ we have $\lim f_\alpha(x) = 0$ and $0 \leqslant f_\alpha(x) \leqslant f_\gamma(x)$ whenever $\alpha \geqslant \gamma$. An element ϕ of A' is called τ-additive if $\phi(f_\alpha) \longrightarrow 0$ whenever $f_\alpha \downarrow 0$. We will say that ϕ is σ-additive if $\phi(f_n) \longrightarrow 0$ for each sequence $\{f_n\}$ in A which decreases to zero. The set of all σ-additive (τ-additive) members of A' will be denoted by $L_\sigma(A)$ ($L_\tau(A)$).

THEOREM 5.2. *Each of the dual spaces $(A, \beta)'$, $(A, \beta_1)'$ and $(A, \beta_F)'$ forms a linear lattice ideal in the Riesz space A'.*

PROOF. This follows easily from the fact that the spaces (A, β), (A, β_1) and (A, β_F) are locally solid.

THEOREM 5.3. *The dual space of the space (A, β) is the space $L_\tau(A)$.*

PROOF. Let $\phi \in A'$ and let $m \in M_\tau(\text{Bo}(Y), E')$ be such that $\phi(f) = \int \hat{f}\, dm$ for all $f \in A$. Suppose ϕ is β-continuous and let $f_\alpha \downarrow 0$. We want to show that $\phi(f_\alpha) \longrightarrow 0$. Without loss of generality we may assume that $\|f_\alpha\| \leqslant 1$ for all α. Let $\epsilon > 0$. For each α, set $Z_\alpha = \{x \in Y : \|\hat{f}_\alpha(x)\| \geqslant \epsilon\}$. Then $Z_\alpha \downarrow Q = \bigcap Z_\alpha$. Since $Q \in \Omega$ we have $|m|(Q) = 0$ by 4.9. Since $|m|(Z_\alpha) \longrightarrow |m|(Q) = 0$, there exists α_0 such that $|m|(Z_\alpha) < \epsilon$ for all $\alpha \geqslant \alpha_0$. Now, for $\alpha \geqslant \alpha_0$, we have

$$|\phi(f_\alpha)| \leqslant \left|\int_{Z_\alpha} f\, dm\right| + \left|\int_{Y-Z_\alpha} \hat{f}\, dm\right|$$
$$\leqslant |m|(Z_\alpha) + \epsilon\|m\| \leqslant \epsilon(1 + \|m\|).$$

This shows that $\phi(f_\alpha) \longrightarrow 0$ and ϕ is τ-additive.

Conversely, assume that ϕ is τ-additive. Let Q in Ω and $0 \leqslant s \in E$, $\|s\| \leqslant 1$. Choose an open set O in Y, $Q \subset O$, such that $|m|(O - Q) < \epsilon$ ($\epsilon > 0$ arbitrary). The collection $D = \{hs : h \in B,\ 0 \leqslant h \leqslant 1,\ \hat{h} = 1 \text{ on } Q \text{ and } \hat{h} = 0 \text{ on } Y - O\}$ is downwards directed to zero. Hence there exists hs in D such that $|\phi(hs)| < \epsilon$. But

$$|\phi(hs)| \geqslant \left|\int_Q s\,dm\right| - \left|\int_{O-Q} \hat{h}s\,dm\right| \geqslant |m(Q)s| - |m|(O - Q) \geqslant |m(Q)s| - \epsilon.$$

Hence $|m(Q)s| \leqslant 2\epsilon$. This proves that $m(Q)s = 0$ for each Q in Ω and each $s \in E$, $\|s\| \leqslant 1$, $s \geqslant 0$. Since E is a lattice, we have $m(Q)s = 0$ for all $s \in E$. Now the regularity of ms and 4.9 complete the proof.

THEOREM 5.4. *The dual of the space* (A, β_1) *is the space* $L_\sigma(A)$.

PROOF. Let $\phi \in A'$ and $m \in M_\tau(\text{Bo}(Y), E')$ be such that $\phi(f) = \int_Y \hat{f}\,dm$ for all f in A. Assume that ϕ is β_1-continuous and let $\{f_n\}$ be a sequence in A that decreases to zero, We want to show that $\phi(f_n) \longrightarrow 0$. We may assume without loss of generality that $\|f_n\| \leqslant 1$ for all n. Let $\epsilon > 0$ and set $Z_n = \{x \in Y : \|\hat{f}_n(x)\| \geqslant \epsilon\}$. Then $Z_n \downarrow \bigcap Z_n = Z$ and $Z \in \Omega_1$. Since $\lim |m|(Z_n) = |m|(Z) = 0$, there exists n_0 such that $|m|(Z_n) < \epsilon$ if $n \geqslant n_0$. Now, if $n \geqslant n_0$, we have

$$|\phi(f_n)| \leqslant \left|\int_{Z_n} \hat{f}_n\,dm\right| + \left|\int_{Y-Z_n} \hat{f}_n\,dm\right| < \epsilon + \epsilon\|m\|.$$

This proves that $\phi(f_n) \longrightarrow 0$ and hence ϕ is σ-additive. Conversely, assume $\phi \in L_\sigma(A)$. Let Z be in Ω_1 and $s \in E$ with $s \geqslant 0$ and $\|s\| \leqslant 1$. Given $\epsilon > 0$ there exists a cozero set V containing Z such that $|m|(V - Z) < \epsilon$. Let $g \in B$, $0 \leqslant g \leqslant 1$, be such that $Z = \hat{g}^{-1}\{0\}$. For each positive integer n, let $V_n = \{x \in Y : \hat{g}(x) < 1/n\} \cap V$. Choose $h_n \in B$, $0 \leqslant h_n \leqslant 1$, $\hat{h}_n = 1$ on Z and $\hat{h} = 0$ on $Y - V_n$. Let $h'_n = h_1 \wedge \cdots \wedge h_n$ and set $f_n = h'_n s$. Then $f_n \downarrow 0$. Hence there exists n such that $|\phi(f_n)| < \epsilon$. But

$$\begin{aligned}|\phi(f_n)| &\geqslant \left|\int_Z \hat{f}_n\,dm\right| - \left|\int_{V-Z} \hat{f}_n\,dm\right| \\ &\geqslant |m(Z)s| - |m|(V - Z) \geqslant |m(Z)s| - \epsilon.\end{aligned}$$

Thus $|m(Z)s| \leqslant 2\epsilon$. This proves that $m(Z)s = 0$. From this it follows that $m(Z)s = 0$ for each $s \in E$ and all Z in Ω_1. Now the result follows from the regularity of $ms|_{\text{Ba}(Y)}$ and from 4.9.

THEOREM 5.5. *Let* τ *be a locally convex Hausdorff topology on* A *for which the positive cone is normal. Then the following assertions are equivalent*:

(1) $(A, \tau)' \subset L_\tau(A)$.

(2) *If* $f_\alpha \downarrow 0$, *then* $f_\alpha \longrightarrow 0$ *in the* τ-*topology*.

PROOF. It is clear that (2) implies (1).

(1) $\longrightarrow$ (2). By Schaefer [10, p. 219], τ is the topology of uniform convergence on the τ-equicontinuous subsets of $(A, \tau)'^{+} = \{\phi \in (A, \tau)' : \phi \geqslant 0\}$. Suppose now that $f_\alpha \downarrow 0$ and let V be a τ-neighborhood of zero. There exists a τ-equicontinuous subset H of $(A, \tau)'^{+}$ such that $H^0 \subset V$. The set H is relatively weakly compact. Every f_α defines a weakly continuous linear functional on $(A, \tau)'$ by $\phi \longrightarrow W_\alpha(f) = \phi(f_\alpha)$. If $\phi \in H$, then $W_\alpha(\phi) \downarrow 0$. Hence $W_\alpha \longrightarrow 0$ uniformly on H by Dini's theorem. It follows that there exists α_0 such that $f_\alpha \in H^0 \subset V$ for all $\alpha \geqslant \alpha_0$. This proves that $f_\alpha^\tau \longrightarrow 0$ and the proof is complete.

We have an analogous theorem for $L_\sigma(A)$ with a similar proof.

THEOREM 5.6. *Let τ be a locally convex Hausdorff topology on A for which the positive cone is normal. The following are equivalent*:

(1) *The space $(A, \tau)'$ is contained in $L_\sigma(A)$.*

(2) *$f_n \downarrow 0$ implies that $f_n \longrightarrow 0$ in the τ-topology.*

COROLLARY 5.7. (1) *$f_n \downarrow 0$ implies that $f_n \longrightarrow 0$ in the β_1-topology.*

(2) *$f_\alpha \downarrow 0$ implies that $f_\alpha \longrightarrow 0$ in the β-topology.*

THEOREM 5.8. *Let τ be any one of the topologies β, β_1, β_F. If W is a τ-neighborhood of zero, then each of the sets $H_1 = \{\phi^+ : \phi \in W^0\}$, $H_2 = \{\phi^- : \phi \in W^0\}$, and $H_3 = \{|\phi| : \phi \in W^0\}$ is τ-equicontinuous, where W^0 is the polar of W in $(A, \tau)'$.*

PROOF. Since (A, τ) is locally solid, τ is the topology of uniform convergence on the τ-equicontinuous subsets of $(A, \tau)'^{+}$. Let W_1 be a solid τ-neighborhood of zero contained in W and let H be a τ-equicontinuous subset of $(A, \tau)'^{+}$ with $H^0 \subset W_1$. Let $f \in H^0$ and $\phi \in W_1^0 \subset H^{00}$. Since W_1 is solid we have

$$|\phi^+(f)| \leqslant \phi^+(|f|) = \sup\{\phi(h) : 0 \leqslant h \leqslant |f|\} \leqslant 1.$$

Thus $\phi^+ \in H^{00}$. This shows that $H_1 \subset H^{00}$. Similarly $H_2, H_3 \subset H^{00}$ and the theorem is proved.

Throughout the remaining part of this paper E will be assumed to be a Banach lattice with a unit element e (e has the property that $-e \leqslant s \leqslant e$ iff $\|s\| \leqslant 1$).

THEOREM 5.9. (1) *Every weakly compact subset of $L_\sigma^+(A)$ is β_1-equicontinuous.*

(2) *Every weakly compact subset of $L_\tau^+(A)$ is β-equicontinuous.*

(3) *β is the topology of uniform convergence on the weakly compact subsets of $L_\tau^+(A)$.*

(4) *β_1 is the topology of uniform convergence on the weakly compact subsets of $L_\sigma^+(A)$.*

PROOF. (1) Let $H \subset L_\sigma^+(A)$ be weakly compact. The set H^0 is convex, balanced and absorbent. Let $r > 0$. Set $\alpha = \sup\{\|\phi\| : \phi \in H\}$ and let Z be in Ω_1. There exists $f \in B$, $0 \leq f \leq 1$, $Z = \hat{f}^{-1}\{0\}$. For each positive integer n, put $Z_n = \{x \in Y : \hat{f}(x) \geq 1/n\}$. Choose $g_n \in B$, $0 \leq g_n \leq 1$, $\hat{g}_n = 1$ on Z and $\hat{g}_n = 0$ on Z_n. Let $h_n = g_1 \wedge \cdots \wedge g_n$. Then $h_n e \downarrow 0$ and hence $\phi(h_n e) \downarrow 0$ for each ϕ in H. By Dini's theorem, $\phi(h_n e) \longrightarrow 0$ uniformly on H. Hence there exists n such that $\phi(h_n e) < 1/(2r)$ for all $\phi \in H$. Let $g = 1 - h_n$ and set $V = \{h \in A : \|gh\| \leq 1/(2a)\}$. If $h \in V \cap U_r$, then $h_n|h| \leq rh_n e$. Hence for $h \in V \cap U_r$ and $\phi \in H$ we have

$$|\phi(h)| \leq \phi(|h|g) + \phi(h_n|h|) \leq \|\phi\|\,\|gh\| + r\phi(h_n e) \leq a(1/2a)) + r(1/2r)) = 1.$$

This shows that $V \cap U_r \subset H^0$. Since this happens for all $r > 0$ and all $Z \in \Omega_1$ we conclude that H^0 is a β_1-neighborhood of zero.

(2) Let $Q \in \Omega$. The set $D = \{g \in B : 0 \leq g \leq 1, \hat{g} = 1 \text{ on } Q\}$ is directed downwards to zero. From here on the proof is similar to that of (1).

(3) If $H \subset L_\tau^+(A)$ is weakly compact, then H^0 is a β-neighborhood of zero by (2). Conversely, let W be a β-neighborhood of zero. Since β is locally solid there exists a β-equicontinuous subset H of $L_\tau^+(A)$ such that $H^0 \subset W$. If H_1 is the weak closure of H in $L_\tau(A)$, then $H_1 \subset L_\tau^+(A)$ and H_1 is weakly compact. Moreover $H_1^0 \subset H^0 \subset W$. This proves (3).

(4) The proof is similar to that of (3).

COROLLARY 5.10. $\beta = \beta_1$ *iff* $L_\tau(A) = L_\sigma(A)$.

Using Dini's theorem and the Alaoglu-Bourbaki theorem (see Köthe [14, p. 248]) we can easily show the following

THEOREM 5.11. (1) *A subset H of $L_\tau^+(A)$ is $\sigma(L_\tau(A), A)$ relatively compact iff $\phi(f_\alpha) \longrightarrow 0$ uniformly on H for each net $\{f_\alpha\}$ in A that decreases to zero.*

(2) *A subset H of $L_\sigma^+(A)$ is $\sigma(L_\sigma(A), A)$ relatively compact iff $\phi(f_n) \longrightarrow 0$ uniformly on H whenever $f_n \downarrow 0$.*

COROLLARY 5.12. *Let $H \subset L_\tau(A)$ $(H \subset L_\sigma(A))$. The following are equivalent*:

(1) *$\{\phi^+ : \phi \in H\}$ and $\{\phi^- : \phi \in H$ are both weakly relatively compact in $L_\tau(A)$ (in $L_\sigma(A)$).*

(2) *$\{|\phi| : \phi \in H\}$ is weakly relatively compact in $L_\tau(A)$ (in $L_\sigma(A)$).*

THEOREM 5.13. *The following are equivalent*:

(1) *(A, β) is a Mackey space.*

(2) *If H is a convex, balanced, weakly compact subset of $L_\tau(A)$, then $\{|\phi| : \phi \in H\}$ is weakly relatively compact in $L_\tau(A)$.*

(3) *If H is a convex, balanced, weakly compact subset of $L_\tau(A)$, then $\{\phi^+ : \phi \in H\}$ and $\{\phi^- : \phi \in H\}$ are both weakly relatively compact in $L_\tau(A)$.*

PROOF. By 5.12, (2) is equivalent to (3).

(1) $\longrightarrow$ (3). Let H be a weakly compact, convex, balanced subset of $L_\tau(A)$. By hypothesis H^0 is a β-neighborhood of zero. By 5.8 the sets $V_1 = \{\phi^+ : \phi \in H^{00}\}$ and $V_2 = \{\phi^- : \phi \in H^{00}\}$ are weakly relatively compact in $L_\tau(A)$. Since $H \subset H^{00}$, (3) follows.

(3) $\longrightarrow$ (1). Let H be a convex balanced weakly compact subset of $L_\tau(A)$. By hypothesis and 5.9, the sets $H_1 = \{\phi^+ : \phi \in H\}$ and $H_2 = \{\phi^- : \phi \in H\}$ are β-equicontinuous. Since $H^0 \supset \frac{1}{2}(H_1^0 \cap H_2^0)$, it follows that H is β-equicontinuous. Hence β is finer than the Mackey topology $m(A; L_\tau(A))$. Thus $\beta = m(A, L_\tau(A))$ since $(A, \beta)' = L_\tau(A)$. This completes the proof.

We have an analogous theorem for the pair $\langle A, L_\sigma(A)\rangle$ and the topology β_1. The proof is similar.

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